Modern University for Technology and Information
Faculty of Engineering
Mathematics (3) Final Exam
Fall 2014, Time allowed 2 hours

Answer the following questions:

(1- a) Use the suitable test to test the following series for convergence

$$(i)\ \sum_{n=1}^{\infty} \frac{1}{n^2+1} \qquad (ii)\ \sum_{n=1}^{\infty} \frac{n}{3^n} \qquad (iii)\ \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$$

(1-b) Given $w = \tan^{-1}(x^3 + y^3)$ Show that $x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = 3\sin w \cos w$

-- **(8 marks)**

(2) Solve the following differential equation

(a) $y' + y\cot x = \sin^2 x$

(b) $(xy - x^2)dy - y^2 dx = 0$

-- **(8 marks)**

(3) Find the general solution of the following differential equation

(a) (a) $(D^2 - 2D + 1)y = \cos 3x$

(b) $(D^2 - 5D + 6)y = e^x \cosh 6x$

-- **(8 marks)**

(4-a) Find $\overline{\nabla}.\overline{F}$ and $\overline{\nabla} \times \overline{F}$ given that $\vec{F} = (2xy + z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$

(4-b) Show that $\vec{\nabla} \times \vec{\nabla}\phi = 0$ for any scalar function $\phi(x, y, z)$.

-- **(8 marks)**

(5- a) Evaluate $\displaystyle\oint_C \frac{z^3 - 3z}{(z-2)}dz$ where C is the circle $|z| = 4$ in the complex plane.

(5- b) Evaluate $\displaystyle\int_{(3,0)}^{(-3,0)} (z^2 - iz)dz$ on the circle $|z| = 3$.

(5 -c) Show that the function $f(z) = \sin x \cosh y + i\cos x \sinh y$ is differentiable at any point $\boldsymbol{z}$.

-- **(8 marks)**

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Model Answer

Answer of question (1)

(a) Test the following series for convergence

$$(i)\sum_{n=1}^{\infty}\frac{1}{n^2+1} \qquad (ii)\sum_{n=1}^{\infty}\frac{n}{3^n} \qquad (iii)\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$$

Answer

(i) Use integral test

$$u_x=\frac{1}{x^2+1} \; and \qquad \int_1^{\infty}\frac{1}{x^2+1}dx=\left[\tan^{-1}x\right]_1^{\infty}=\tan^{-1}\infty-\tan^{-1}1=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$$

The integral exist then the series $\sum_{n=1}^{\infty}\frac{1}{n^2+1}$ convergent series

(ii) Use ratio test then

$$u_n=\frac{n}{3^n},\quad u_{n+1}=\frac{n+1}{3^{n+1}},\quad \frac{u_{n+1}}{u_n}=\frac{(n+1)}{3^{n+1}}.\frac{3^n}{n}=\frac{n+1}{3n}$$

and $\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim\limits_{n\to\infty}\frac{n+1}{3n}=\frac{1}{3}<1$

The series convergent series

(iii) $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$ is geometric series with ratio $\mathbf{\frac{2}{3}<1}$ which show that is convergent series

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(b) Given $w=\tan^{-1}(x^3+y^3)$ Show that $x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}=3\sin w\cos w$

$w=\tan^{-1}(x^3+y^3)$ then $\quad \tan w=(x^3+y^3)$

Differentiate with respect to x $\quad \sec^2 w\frac{\partial w}{\partial x}=3x^2 \qquad (1)$

Differentiate with respect to y $\quad \sec^2 w\frac{\partial w}{\partial y}=3y^2 \qquad (2)$

Multiply (1) by x and (2) by y we have

$x\sec^2 w\frac{\partial w}{\partial x}=3x^3 \qquad (3)$

$y\sec^2 w\frac{\partial w}{\partial y}=3y^3 \qquad (4)$

Add (3) and (4)

$$x\sec^2 w\frac{\partial w}{\partial x}+y\sec^2 w\frac{\partial w}{\partial y}=3x^3+3y^3=3(x^3+y^3)=3\tan w$$

Divided by $\sec^2 w$ $\qquad x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}=\frac{3\tan w}{\sec^2 w}=3\sin w\cos w \quad R.T.P$

Answer of question (2)

(a) $\boldsymbol{y'+y\cot x=\sin^2 x}$

The equation is a linear and $\boldsymbol{P(x)=\cot x,\quad Q(x)=\sin^2 x}$

We can determine the integrating factor

$\boldsymbol{\int P(x)dx=\int \cot x dx=\ln\sin x} \qquad \boldsymbol{\therefore \mu=e^{\ln\sin x}=\sin x}$

multiply the equation by $\boldsymbol{\sin x}$

$$y'\sin x+y\sin x\cot x=\sin^3 x$$

$$y'\sin x+y\sin x\frac{\cos x}{\sin x}=\sin^3 x$$

$$y'\sin x+y\cos x=\sin^3 x$$

Which is exact differential equation and the left side is the derivative of $\boldsymbol{y\sin x}$

$\boldsymbol{\therefore d(y\sin x)=\sin^3 x}$ and by integration

$$y\sin x=\int \sin^3 x\,dx=\int\sin^2 x\sin x\,dx=\int(1-\cos^2 x)\sin x\,dx=\int(\sin x dx-\cos^2 x\sin x\,dx)$$

$$y\sin x=-\cos x+\frac{1}{3}\cos^3 x+C$$

$$\therefore \boxed{y=-\cot x+\frac{1}{3}\frac{\cos^2 x}{\sin x}+\frac{C}{\sin x}}$$

is the general solution of the given equation.

(b) $\boldsymbol{(xy-x^2)dy-y^2dx=0}$

$\boldsymbol{M(x,y),N(x,y)}$ are homogeneous of the same degree(second degree)

let $\boldsymbol{y=ux}$ $\quad\boldsymbol{\therefore dy=udx+xdu}$ $\quad$ substitute in the differential equation we have

$$(x^2u - x^2)(udx + xdu) - x^2u^2dx = 0$$

$$x^2(u-1)(udx + xdu) - x^2u^2dx = 0$$

Divided by x^2 and separate the variable

$$(u-1)(udx + xdu) - u^2dx = 0$$

$$(u-1)udx + (u-1)xdu - u^2dx = 0$$

$$\left[(u-1)u - u^2\right]dx + (u-1)xdu = 0$$

$$-udx + (u-1)xdu = 0$$

$$-\frac{dx}{x} + \frac{(u-1)}{u}du = 0$$

$-\dfrac{dx}{x} + (1 - \dfrac{1}{u})du = 0$ by integration we have

$$\boxed{-\ln x + u - \ln u = \ln c}$$

$$\ln x + \ln u + \ln c = u$$

$$\ln cxu = u \qquad \Rightarrow \qquad \boxed{\ln cy = \frac{y}{x}}$$

<u>Another solution</u>

$$(xy - x^2)dy - y^2dx = 0 \quad \Rightarrow \quad xydy - x^2dy - y^2dx = 0$$

$$xydy - y^2dx - x^2dy = 0$$

$$y(xdy - ydx) - x^2dy = 0 \quad \Rightarrow \quad \frac{(xdy - ydx)}{x^2} - \frac{1}{y}dy = 0$$

$d(\dfrac{y}{x}) - \dfrac{1}{y}dy = 0$ integrate $\boxed{\dfrac{y}{x} - \ln y = \ln C}$

$$\frac{y}{x} = \ln C + \ln y = \ln Cy \quad \Rightarrow \quad \boxed{y = xe^{Cy}}$$

Answer of question (3)

(a) $(D^2 - 2D + 1)y = \cos 3x$ The characteristic equation is

$$m^2 - 2m + 1 = 0 \quad \Rightarrow \quad (m-1)^2 = 0$$

$\therefore m_1 = 1,\ m_2 = 1$ and $\boxed{y_C = (C_1 + C_2x)e^x}$

$$y_P=\frac{1}{D^2-2D+1}\cos 3x=\frac{1}{-9-2D+1}\cos 3x=\frac{-1}{2}.\frac{1}{(D+4)}\cos 3x=\frac{-1}{2}.\frac{(D-4)}{(D^2-16)}\cos 3x$$

$$=\frac{-1}{2}.\frac{(D-4)}{(-9-16)}\cos 3x=\frac{1}{50}(D-4)\cos 3x=\frac{1}{50}.(-3\sin 3x-4\cos 3x)$$

$$\therefore \boxed{y_G(x)=(C_1+C_2x)e^x-\frac{1}{50}.(3\sin 3x+4\cos 3x)}$$

(b) $(D^2-5D+6)y=e^x\cosh 6x$ The characteristic equation is

$m^2-5m+6=0$ then $(m-2)(m-3)=0\rightarrow m=2,3$ and $y_c=C_1e^{3x}+C_2e^{2x}$ and

$$y_p=\frac{1}{D^2-5D+6}e^x\cosh 6x=e^x\frac{1}{(D+1)^2-5(D+1)+6}\cosh 6x$$

$$=e^x\frac{1}{D^2+2D+1-5D-5+6}\cosh 6x=e^x\frac{1}{D^2-3D+2}\cosh 6x$$

$$=e^x\frac{1}{6^2-3D+2}\cosh 6x=e^x\frac{1}{38-3D}\cosh 6x=e^x\frac{(38-3D)}{38^2-9D^2}\cosh 6x$$

$$=\frac{e^x}{38^2-9(6)^2}(38\cosh 6x-18\sinh 6x)$$

Then the general solution in the form

$$\boxed{y_G=C_1e^{3x}+C_2e^{2x}+\frac{e^x}{38^2-9(6)^2}(38\cosh 6x-18\sinh 6x)}$$

<u>Another solution</u>

$$y_p=\frac{1}{D^2-5D+6}e^x\cosh 6x=\frac{1}{D^2-5D+6}e^x\left(\frac{e^{6x}+e^{-6x}}{2}\right)$$

$$=\frac{1}{2}\frac{1}{D^2-5D+6}\left(e^{7x}+e^{-5x}\right)=\frac{1}{2}\left(\frac{1}{D^2-5D+6}e^{7x}+\frac{1}{D^2-5D+6}e^{-5x}\right)$$

$$=\frac{1}{2}\left(\frac{1}{7^2-5(7)+6}e^{7x}+\frac{1}{(-5)^2-5(5)+6}e^{-5x}\right)=\frac{1}{2}\left(\frac{1}{20}e^{7x}+\frac{1}{6}e^{-5x}\right)$$

Then the general solution in the form

$$\boxed{y_G=C_1e^{3x}+C_2e^{2x}+\frac{1}{2}\left(\frac{1}{20}e^{7x}+\frac{1}{6}e^{-5x}\right)}$$

Answer of question (4)

(4-a) Find $\overline{\nabla}.\overline{F}$ *and* $\overline{\nabla}\times\overline{F}$ given that $\vec{F}=(2xy+z^3)\vec{i}+x^2\vec{j}+3xz^2\vec{k}$

Answer

$$div\ \vec{F}=\nabla\cdot\vec{F}=\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}=\frac{\partial(2xy+z^3)}{\partial x}+\frac{\partial x^2}{\partial y}+\frac{\partial 3xz^2}{\partial z}=\boxed{2y+6xz}$$

$$(\vec{\nabla}\times\vec{F})=[(\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k})\times(x^2y\vec{i}-2xz\overline{j}+2yz\vec{k})]=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ (2xy+z^3) & x^2 & 3xz^2\end{vmatrix}=0$$

(b) Show that $\vec{\nabla}\times\vec{\nabla}\phi=0$ for any scalar function $\phi(x,y,z)$.

$$\vec{\nabla}\times(\vec{\nabla}\phi)=\vec{\nabla}\times\left(\frac{\partial\phi}{\partial x}\vec{i}+\frac{\partial\phi}{\partial y}\vec{j}+\frac{\partial\phi}{\partial z}\vec{k}\right)=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z}\end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\right)-\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial y}\right)\right]\vec{i}+\left[\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial x}\right)-\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial z}\right)\right]\vec{j}+\left[\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right)-\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right)\right]\vec{k}$$

$$=\left[\frac{\partial^2\phi}{\partial y\partial z}-\frac{\partial^2\phi}{\partial z\partial y}\right]\vec{i}+\left[\frac{\partial^2\phi}{\partial z\partial x}-\frac{\partial^2\phi}{\partial x\partial z}\right]\vec{j}+\left[\frac{\partial^2\phi}{\partial x\partial y}-\frac{\partial^2\phi}{\partial y\partial x}\right]\vec{k}=0$$

Answer of question (5)

(5- a) Evaluate $\oint_C \frac{z^3-3z}{(z-2)}dz$ where C is the circle $|z|=4$ in the complex plane.

Since $z=2$ inside the circle $|z|=4$ and $\frac{f(z)}{z-z_0}=\frac{z^3-3z}{(z-2)}$ *then* $z_0=2,\ f(z)=z^3-3z$

$f(z_0)=(2)^3-3(2)=2$ by using Cauchy's integral Formula

$$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0) \quad \text{then} \quad \oint_C \frac{z^3-3z}{z-2}\,dz = (2\pi i)\, f(2) = 4\pi i$$

(b) Evaluate $\int_{(3.0)}^{(-3,0)} (z^2 - iz)dz$ on the circle $|z| = 3$.

Answer

Use the exponential form for the complex number

$$z = 3e^{i\theta} \text{ then } d\theta = 3e^{i\theta}d\theta \;\; and \;\; \bar{z} = 3e^{-i\theta}$$

$$\int_{(3,0)}^{(-3,0)} (z^2 - iz)dz = \int_0^\pi (9e^{2i\theta} - i3ie^{i\theta})3ie^{i\theta}d\theta = \int_0^\pi (9e^{2i\theta} + 3e^{i\theta})3ie^{i\theta}d\theta$$

$$= 3i\int_0^\pi (9e^{3i\theta} + 3e^{2i\theta})d\theta = 3i\left[9\frac{e^{3i\theta}}{3i} + 3\frac{e^{2i\theta}}{2i}\right]$$

$$= 3i\left[9\frac{e^{3i\theta}}{3i} + 3\frac{e^{2i\theta}}{2i}\right]_0^\pi = 3[-6] = -18$$

(5 -c) Show that the function $f(z) = \sin x \cosh y + i \cos x \sinh y$ is differentiable at any point z.

$$f(z) = \sin x \cosh y + i \cos x \sinh y$$

$$u = \sin x \cosh y \qquad v = \cos x \sinh y$$

$$\frac{\partial u}{\partial x} = \cos x \cosh y \qquad \frac{\partial v}{\partial x} = -\sin x \sinh y$$

$$\frac{\partial u}{\partial y} = \sin x \sinh y \qquad \frac{\partial v}{\partial y} = \cos x \cosh y$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \cos x \cosh y \;\; and \;\; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \sin x \sinh y$$

This function satisfies Cauchy Riemann’s equations which indicates that is differentiable at any point z.